Consumer Theory

Assumption 1.1: *We will always assume that* X *is a* ***closed and convex set***

$x \succsim y$: *bundle* x *is at least as good as the bundle* y

Axiom 1.1: **Completeness**: $\forall x, y \in X \Rightarrow x \succsim y$ or $y \succsim x$

Axiom 1.2: **Transitivity:** $\forall x, y, z \in X$, if $x \succsim y$ and $y \succsim z \Rightarrow x \succsim z$

Definition 1.1: *The relation* $\succsim$ *on the consumption set X is called* ***a preference relation*** *if it satisfies Axioms 1.1, 1.2 and 1.3.*

DEFINITION 1.2 ***Strict Preference Relation***

The binary relation $\succ$ *on the consumption set* X *is defined as follows:*

$$\mathbf{x}^1 \succ \mathbf{x}^2 \quad \textit{if and only if} \quad \mathbf{x}^1 \succsim \mathbf{x}^2 \quad \textit{and} \quad \mathbf{x}^2 \not\succsim \mathbf{x}^1.$$

The relation $\succ$ *is called the strict preference relation induced by* $\succsim$*, or simply the strict preference relation when* $\succsim$ *is clear. The phrase* $\mathbf{x}^1 \succ \mathbf{x}^2$ *is read, '*$\mathbf{x}^1$ *is strictly preferred to* $\mathbf{x}^2$*'.*

DEFINITION 1.3 ***Indifference Relation***

The binary relation $\sim$ *on the consumption set* X *is defined as follows:*

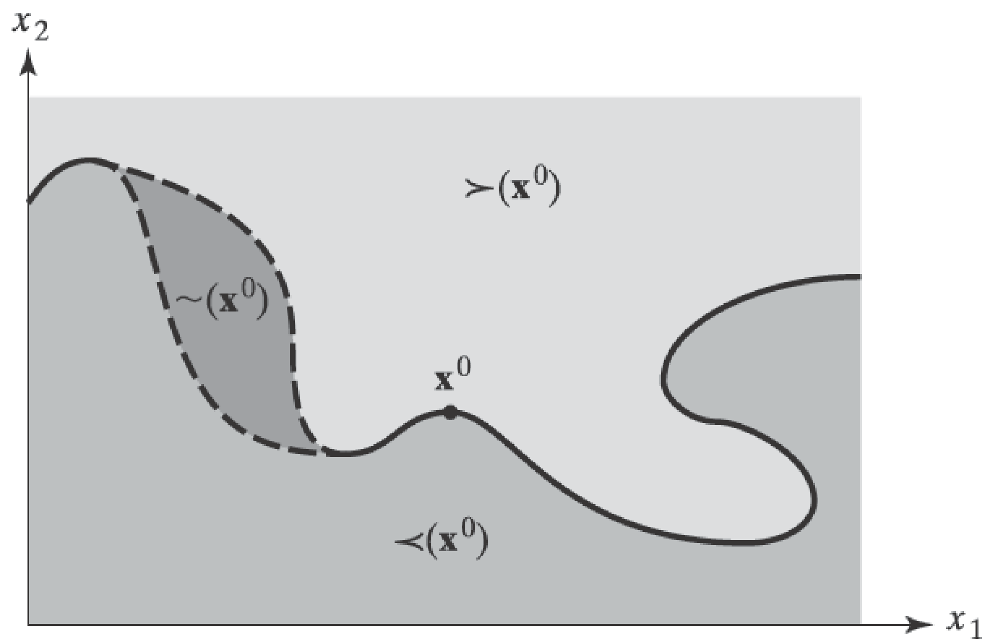
$$\mathbf{x}^1 \sim \mathbf{x}^2 \quad \textit{if and only if} \quad \mathbf{x}^1 \succsim \mathbf{x}^2 \quad \textit{and} \quad \mathbf{x}^2 \succsim \mathbf{x}^1.$$

The relation $\sim$ *is called the indifference relation induced by* $\succsim$*, or simply the indifference relation when* $\succsim$ *is clear. The phrase* $\mathbf{x}^1 \sim \mathbf{x}^2$ *is read, '*$\mathbf{x}^1$ *is indifferent to* $\mathbf{x}^2$*'.*

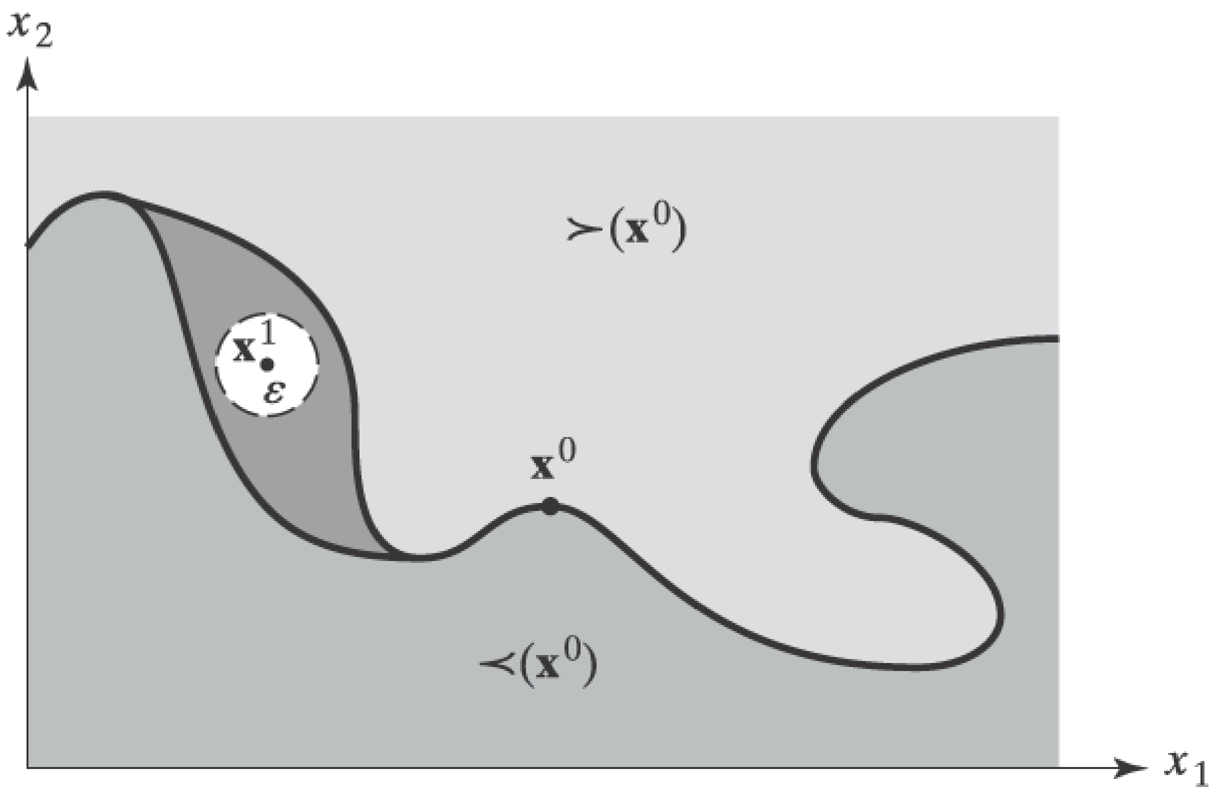
DEFINITION 1.4 ***Sets in X Derived from the Preference Relation***

Let $\mathbf{x}^0$ be any point in the consumption set, X. Relative to any such point, we can define the following subsets of X:

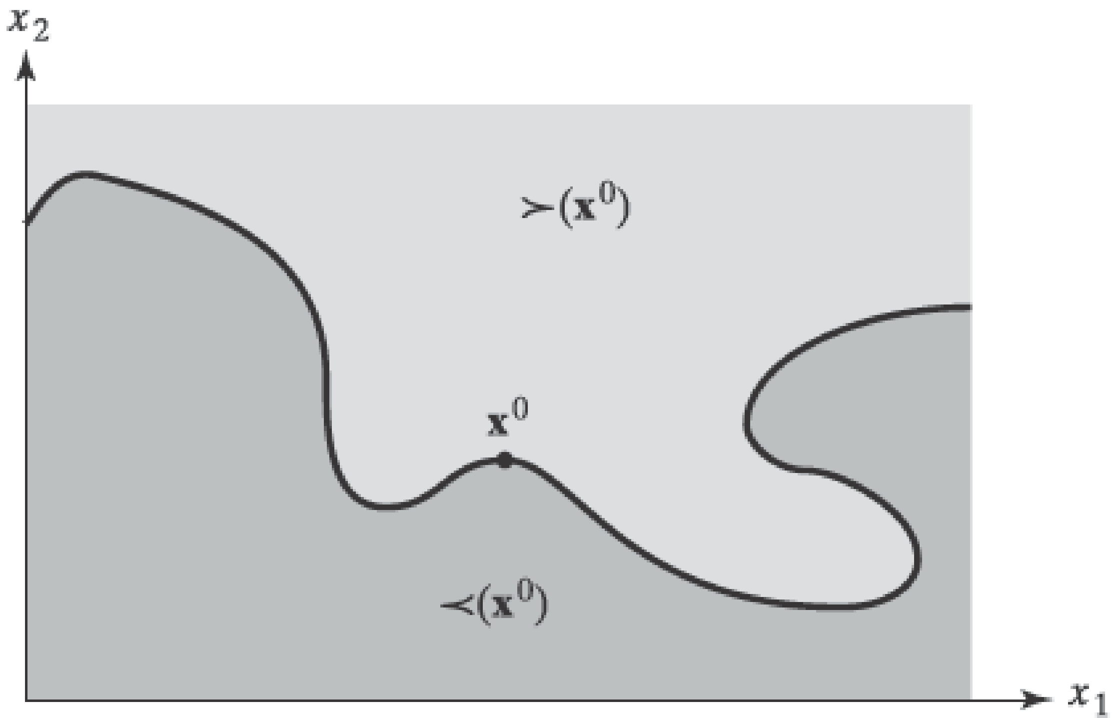
1. $\succsim(\mathbf{x}^0) \equiv \{\mathbf{x} \mid \mathbf{x} \in X, \mathbf{x} \succsim \mathbf{x}^0\}$, *called the 'at least as good as' set.*
2. $\precsim(\mathbf{x}^0) \equiv \{\mathbf{x} \mid \mathbf{x} \in X, \mathbf{x}^0 \succsim \mathbf{x}\}$, *called the 'no better than' set.*
3. $\prec(\mathbf{x}^0) \equiv \{\mathbf{x} \mid \mathbf{x} \in X, \mathbf{x}^0 \succ \mathbf{x}\}$, *called the 'worse than' set.*
4. $\succ(\mathbf{x}^0) \equiv \{\mathbf{x} \mid \mathbf{x} \in X, \mathbf{x} \succ \mathbf{x}^0\}$, *called the 'preferred to' set.*
5. $\sim(\mathbf{x}^0) \equiv \{\mathbf{x} \mid \mathbf{x} \in X, \mathbf{x} \sim \mathbf{x}^0\}$, *called the 'indifference' set.*



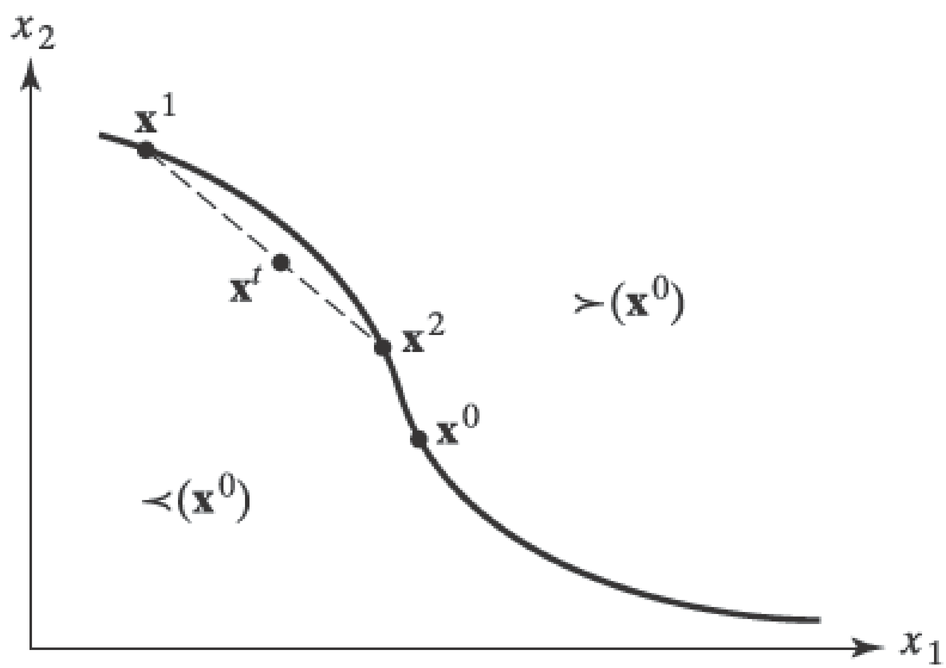
<u>Axiom 1.3</u> Continuity: $\forall y_0 \in X$, *the sets* $\{x : x \succsim y_0\}$ *and* $\{x : y_0 \succsim x\}$ *are closed sets. It follows that* $\{x : x \succ y_0\}$ *and* $\{x : y_0 \succ x\}$ *are open sets.*



AXIOM 4′: Local Non-satiation. *For all* $\mathbf{x}^0 \in \mathbb{R}^n_+$, *and for all* $\varepsilon > 0$, *there exists some* $\mathbf{x} \in B_\varepsilon(\mathbf{x}^0) \cap \mathbb{R}^n_+$ *such that* $\mathbf{x} \succ \mathbf{x}^0$.



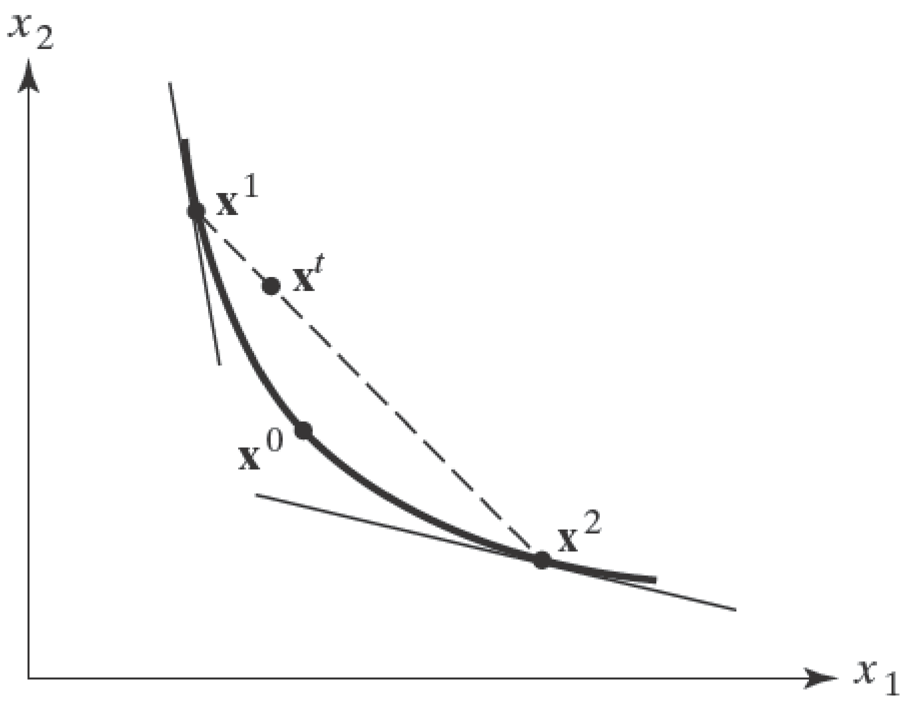
AXIOM 4: Strict Monotonicity. *For all* $\mathbf{x}^0, \mathbf{x}^1 \in \mathbb{R}^n_+$, *if* $\mathbf{x}^0 \geq \mathbf{x}^1$ *then* $\mathbf{x}^0 \succsim \mathbf{x}^1$, *while if* $\mathbf{x}^0 \gg \mathbf{x}^1$, *then* $\mathbf{x}^0 \succ \mathbf{x}^1$.



AXIOM 5′: Convexity. *If* $\mathbf{x}^1 \succsim \mathbf{x}^0$, *then* $t\mathbf{x}^1 + (1-t)\mathbf{x}^0 \succsim \mathbf{x}^0$ *for all* $t \in [0, 1]$.

A slightly stronger version of this is the following:

AXIOM 5: Strict Convexity. *If* $\mathbf{x}^1 \neq \mathbf{x}^0$ *and* $\mathbf{x}^1 \succsim \mathbf{x}^0$, *then* $t\mathbf{x}^1 + (1-t)\mathbf{x}^0 \succ \mathbf{x}^0$ *for all* $t \in (0, 1)$.



Συνάρτηση Χρησιμότητας (Utility function)

DEFINITION 1.5 ***A Utility Function Representing the Preference Relation*** $\succsim$

A real-valued function $u\colon \mathbb{R}^n_+ \to \mathbb{R}$ *is called a utility function representing the preference relation* $\succsim$*, if for all* $\mathbf{x}^0, \mathbf{x}^1 \in \mathbb{R}^n_+$, $u(\mathbf{x}^0) \geq u(\mathbf{x}^1) \Longleftrightarrow \mathbf{x}^0 \succsim \mathbf{x}^1$.

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THEOREM 1.1 ***Existence of a Real-Valued Function Representing the Preference Relation $\succsim$***

If the binary relation $\succsim$ is complete, transitive, continuous, and strictly monotonic, there exists a continuous real-valued function, $u\colon \mathbb{R}^n_+ \to \mathbb{R}$, which represents $\succsim$.

DEFINITION 1.5 ***A Utility Function Representing the Preference Relation*** $\succsim$

A real-valued function $u\colon \mathbb{R}^n_+ \rightarrow \mathbb{R}$ *is called a utility function representing the preference relation* $\succsim$*, if for all* $\mathbf{x}^0, \mathbf{x}^1 \in \mathbb{R}^n_+$, $u(\mathbf{x}^0) \geq u(\mathbf{x}^1) \Longleftrightarrow \mathbf{x}^0 \succsim \mathbf{x}^1$.

THEOREM 1.1 ***Existence of a Real-Valued Function Representing the Preference Relation*** $\succsim$

If the binary relation $\succsim$ *is complete, transitive, continuous, and strictly monotonic, there exists a continuous real-valued function,* $u\colon \mathbb{R}^n_+ \rightarrow \mathbb{R}$*, which represents* $\succsim$.

THEOREM 1.2 ***Invariance of the Utility Function to Positive Monotonic Transforms***

Let $\succsim$ *be a preference relation on* $\mathbb{R}^n_+$ *and suppose* $u(\mathbf{x})$ *is a utility function that represents it. Then* $v(\mathbf{x})$ *also represents* $\succsim$ *if and only if* $v(\mathbf{x}) = f(u(\mathbf{x}))$ *for every* $\mathbf{x}$*, where* $f\colon \mathbb{R} \rightarrow \mathbb{R}$ *is strictly increasing on the set of values taken on by* u.

THEOREM 1.3 ***Properties of Preferences and Utility Functions***

Let $\succsim$ be represented by u: $\mathbb{R}^n_+ \rightarrow \mathbb{R}$. Then:

1. *$u(\mathbf{x})$ is strictly increasing if and only if $\succsim$ is strictly monotonic.*
2. *$u(\mathbf{x})$ is quasiconcave if and only if $\succsim$ is convex.*
3. *$u(\mathbf{x})$ is strictly quasiconcave if and only if $\succsim$ is strictly convex.*

$$u(x_1, x_2) = x_1^{1/4} x_2^{1/4}$$

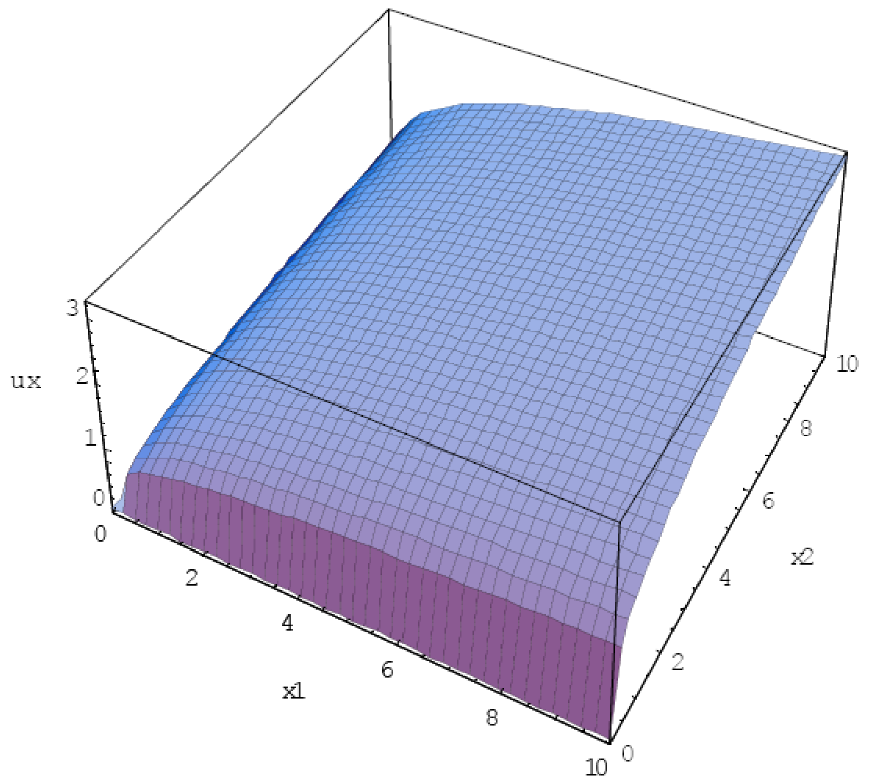


Figure 3.4: Function $u(x)$

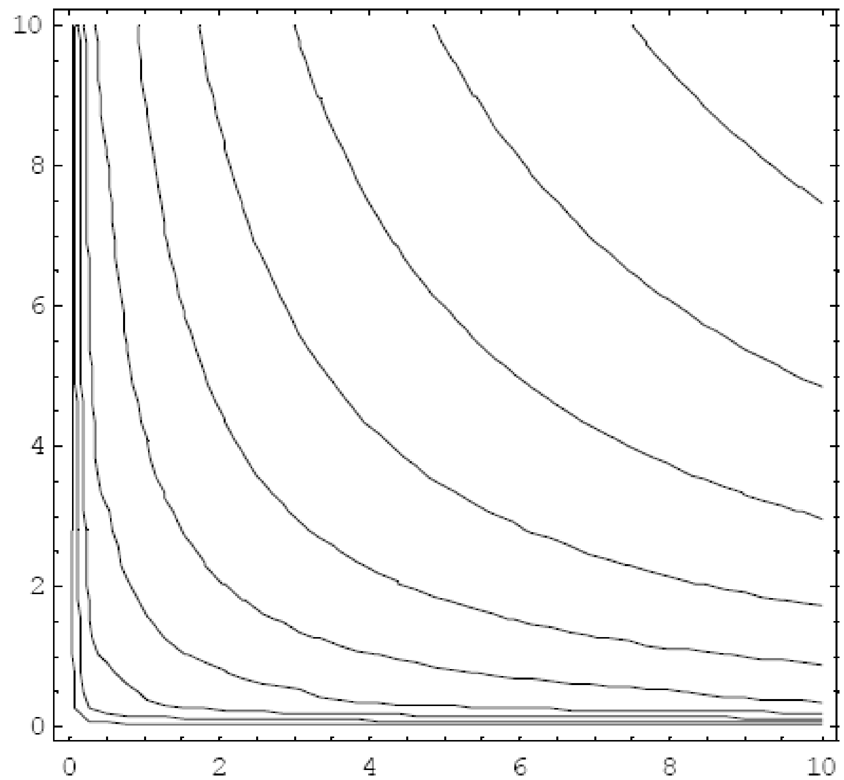


Figure 3.5: Level sets of $u(x)$

$$v(x_1, x_2) = x_1^{3/2} x_2^{3/2}$$

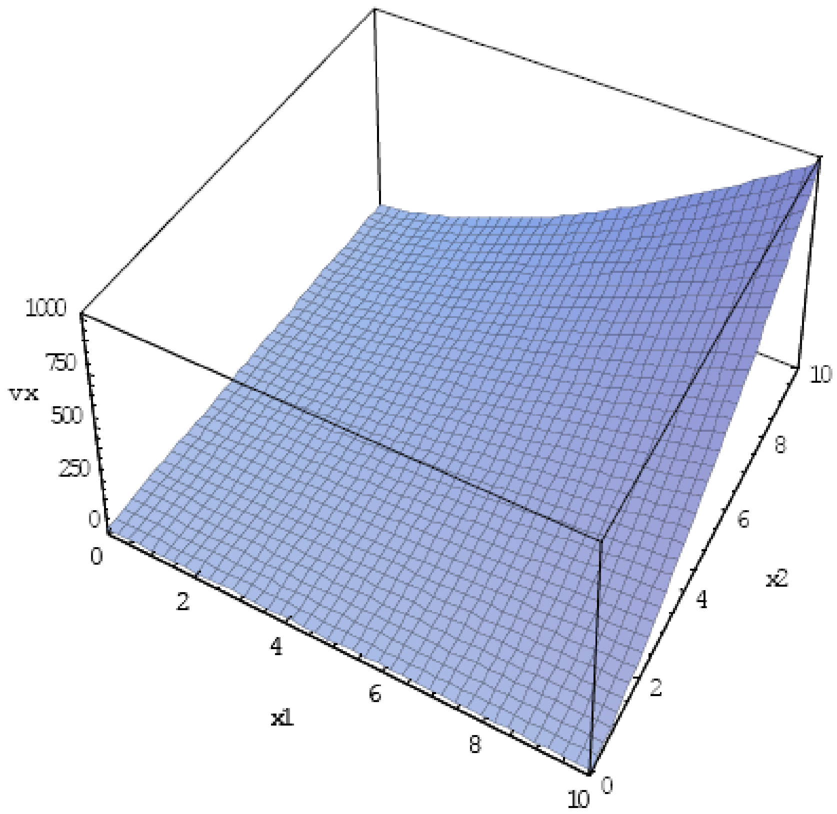


Figure 3.6: Function $v(x)$

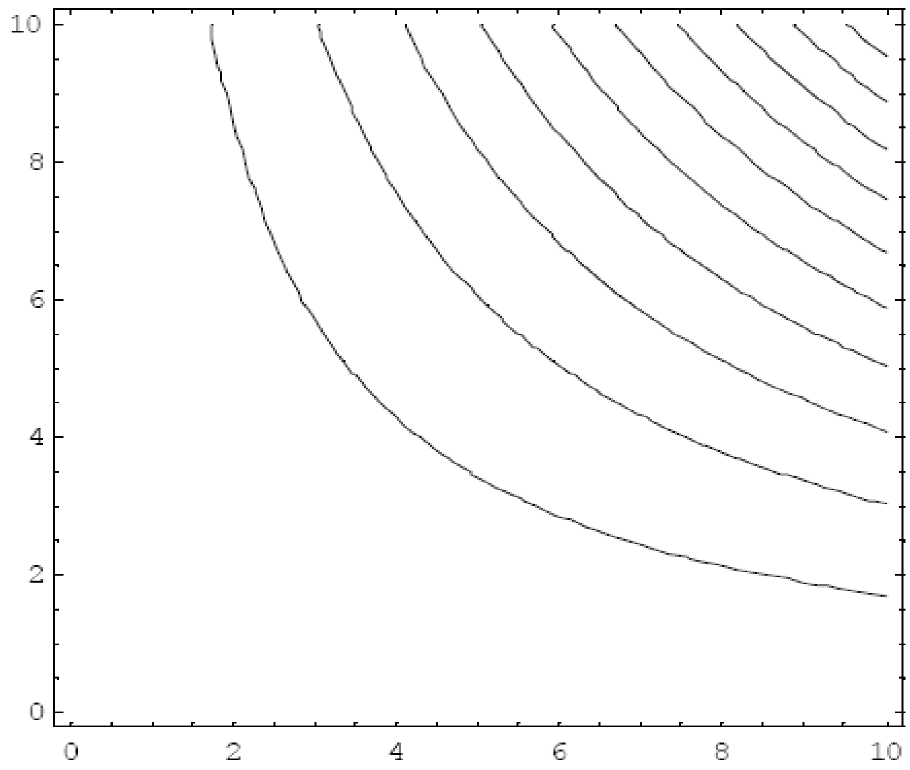


Figure 3.7: Level sets of $v(x)$.

Marginal Rate of Substitution

$$MRS_{ij}(x) = \frac{dx_j}{dx_i} = -\frac{\partial u(x)/\partial x_i}{\partial u(x)/\partial x_j} = -\frac{\text{marginal utility of good i}}{\text{marginal utility of good j}}$$

Utility Maximization Problem

$$B = \{x \setminus x \in \mathbb{R}^n_+ : px \leq w\}$$ **Budget Set**

Utility Maximization Problem

Utility Maximization Problem

ASSUMPTION 1.2 ***Consumer Preferences***

The consumer's preference relation $\succsim$ is complete, transitive, continuous, strictly monotonic, and strictly convex on $\mathbb{R}^n_+$. Therefore, by Theorems 1.1 and 1.3 it can be represented by a real-valued utility function, u, that is continuous, strictly increasing, and strictly quasiconcave on $\mathbb{R}^n_+$.

Utility Maximization Problem

THEOREM 1.5 ***Differentiable Demand***

Let $\mathbf{x}^* \gg \mathbf{0}$ *solve the consumer's maximisation problem at prices* $\mathbf{p}^0 \gg \mathbf{0}$ *and income* $y^0 > 0$*. If*

- u *is twice continuously differentiable on* $\mathbb{R}^n_{++}$*,*
- $\partial u(\mathbf{x}^*)/\partial x_i > 0$ *for some* $i = 1, \ldots, n$*, and*
- *the bordered Hessian of* u *has a non-zero determinant at* $\mathbf{x}^*$*,*

then $\mathbf{x}(\mathbf{p}, y)$ *is differentiable at* $(\mathbf{p}^0, y^0)$*.*

THEOREM 1.6 ***Properties of the Indirect Utility Function***

If $u(\mathbf{x})$ is continuous and strictly increasing on $\mathbb{R}^n_+$, then $v(\mathbf{p}, y)$ defined in (1.12) is

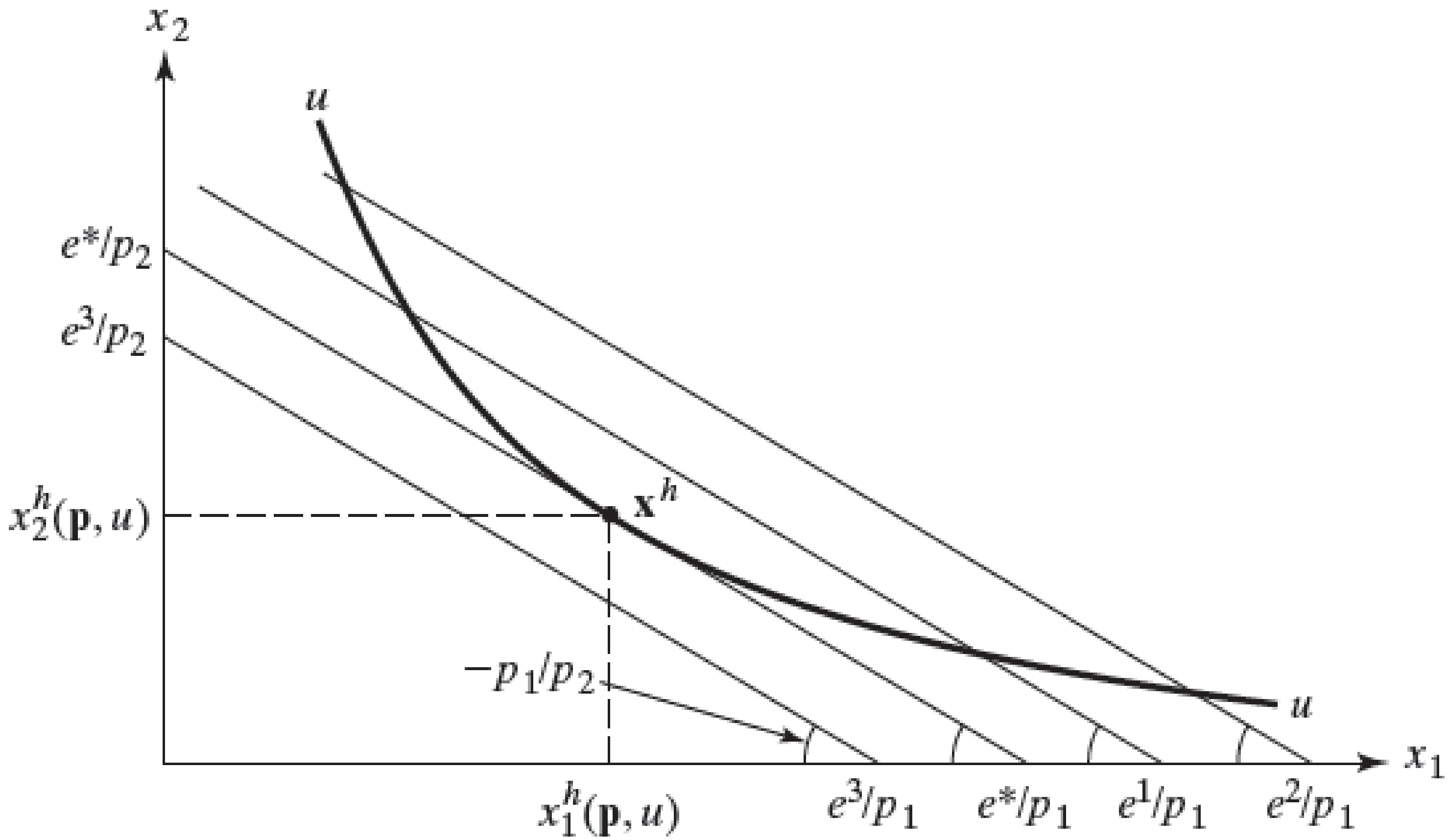
1. *Continuous on $\mathbb{R}^n_{++} \times \mathbb{R}_+$,*
2. *Homogeneous of degree zero in $(\mathbf{p}, y)$,*
3. *Strictly increasing in y,*
4. *Decreasing in $\mathbf{p}$,*
5. *Quasiconvex in $(\mathbf{p}, y)$.*

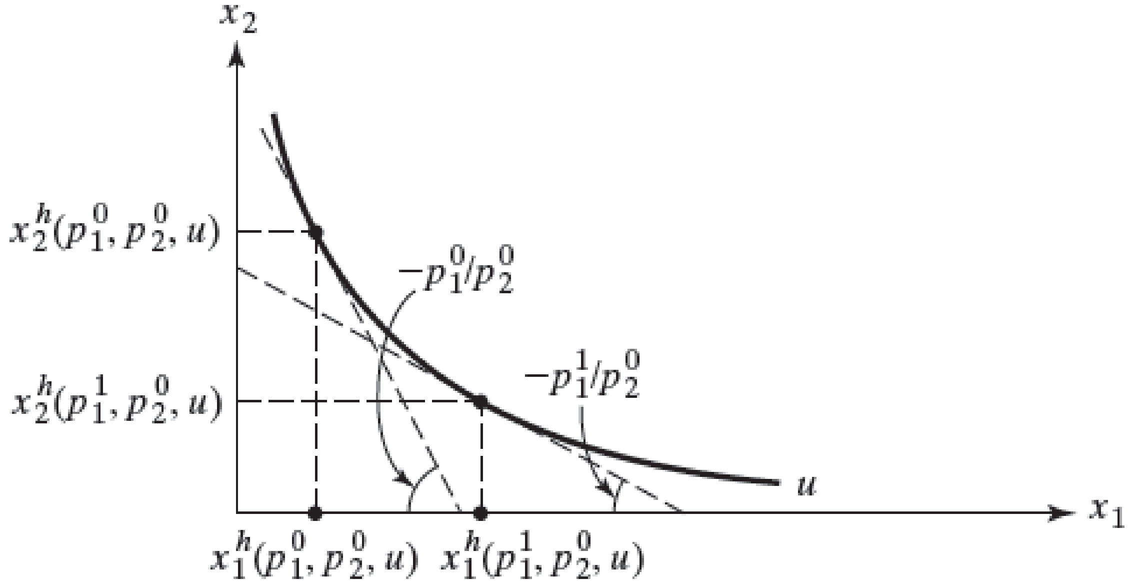
Moreover, it satisfies

6. *Roy's identity: If $v(\mathbf{p}, y)$ is differentiable at $(\mathbf{p}^0, y^0)$ and $\partial v(\mathbf{p}^0, y^0)/\partial y \neq 0$, then*

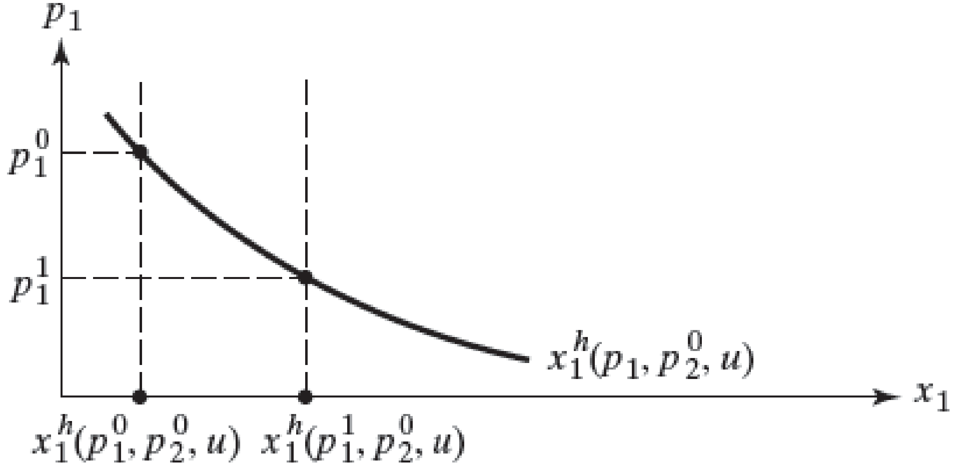
$$x_i(\mathbf{p}^0, y^0) = -\frac{\partial v(\mathbf{p}^0, y^0)/\partial p_i}{\partial v(\mathbf{p}^0, y^0)/\partial y}, \qquad i = 1, \ldots, n.$$

Expenditure function




x_2
$x_2^h(p_1^0, p_2^0, u)$
$x_2^h(p_1^1, p_2^0, u)$
$-p_1^0/p_2^0$
$-p_1^1/p_2^0$
u
x_1
$x_1^h(p_1^0, p_2^0, u)$
$x_1^h(p_1^1, p_2^0, u)$

(a)


p_1
p_1^0
p_1^1
$x_1^h(p_1, p_2^0, u)$
x_1
$x_1^h(p_1^0, p_2^0, u)$
$x_1^h(p_1^1, p_2^0, u)$

(b)

THEOREM 1.7 Properties of the Expenditure Function

If u(.) is continuous and strictly increasing, then $e(p,u)$ *is:*

1. *continuous in p and u*
2. *strictly increasing in u*
3. *increasing in p*
4. *homogeneous of degree one in p*
5. *concave in p*

If in addition, u(.) is strictly quasiconcave (unique solution) we have:

6. *Shephard's lemma:* $e(p,u)$ *is differentiable in p at* (p^0,u^0) *with* $p^0 \gg 0$ *and*

$$\frac{\partial e(p^0,u^0)}{\partial p_i} = x_i^h(p^0,u^0)$$

THEOREM 1.8 Relation between UMP and EMP

Suppose that u(.) is a continuous, strictly increasing utility function and that the price vector is $p \gg 0$*. We have:*

(i) *If* x^* *is optimal in the UMP when income is* $w > 0$ *then* x^* *is optimal in the EMP when the required utility level is* $u(x^*) = v(p, w)$*. Moreover, the minimized expenditure level in this EMP is exactly w*

(ii) *If* x^* *is optimal in the EMP when utility level is u > u(0) then* x^* *is optimal in the UMP when* $w = e(p,u) = px^*$*. Moreover, the maximized utility level in this UMP is exactly u.*

THEOREM 1.8 Some important identities

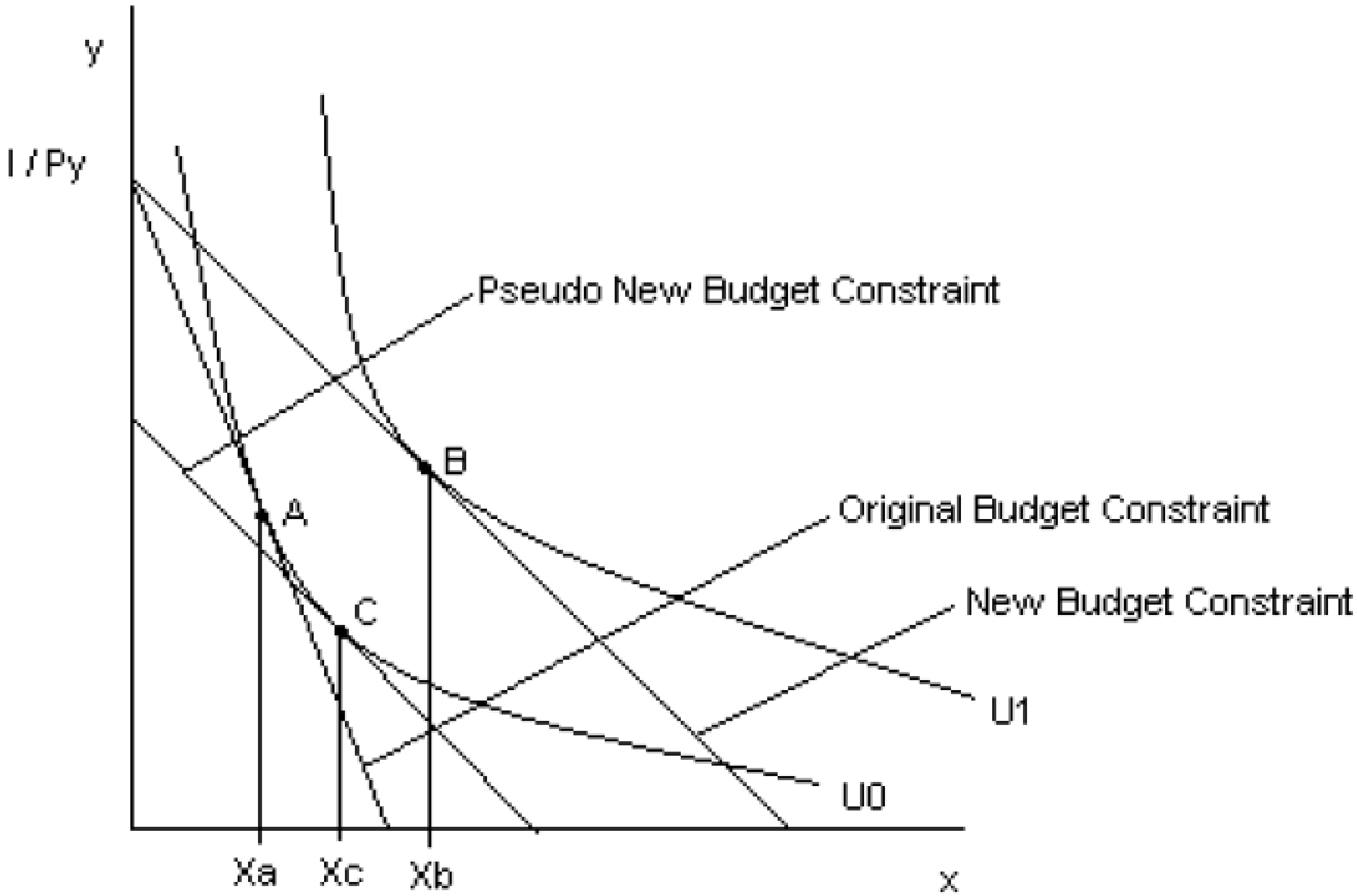
Let $v(p,w)$ *and* $e(p,u)$ *be the indirect utility function and expenditure function for some consumer whose utility function is continuous and strictly increasing. Then for all* $p >> 0$

1. $e(p,v(p,w)) = w$
2. $v(p,e(p,u)) = u$

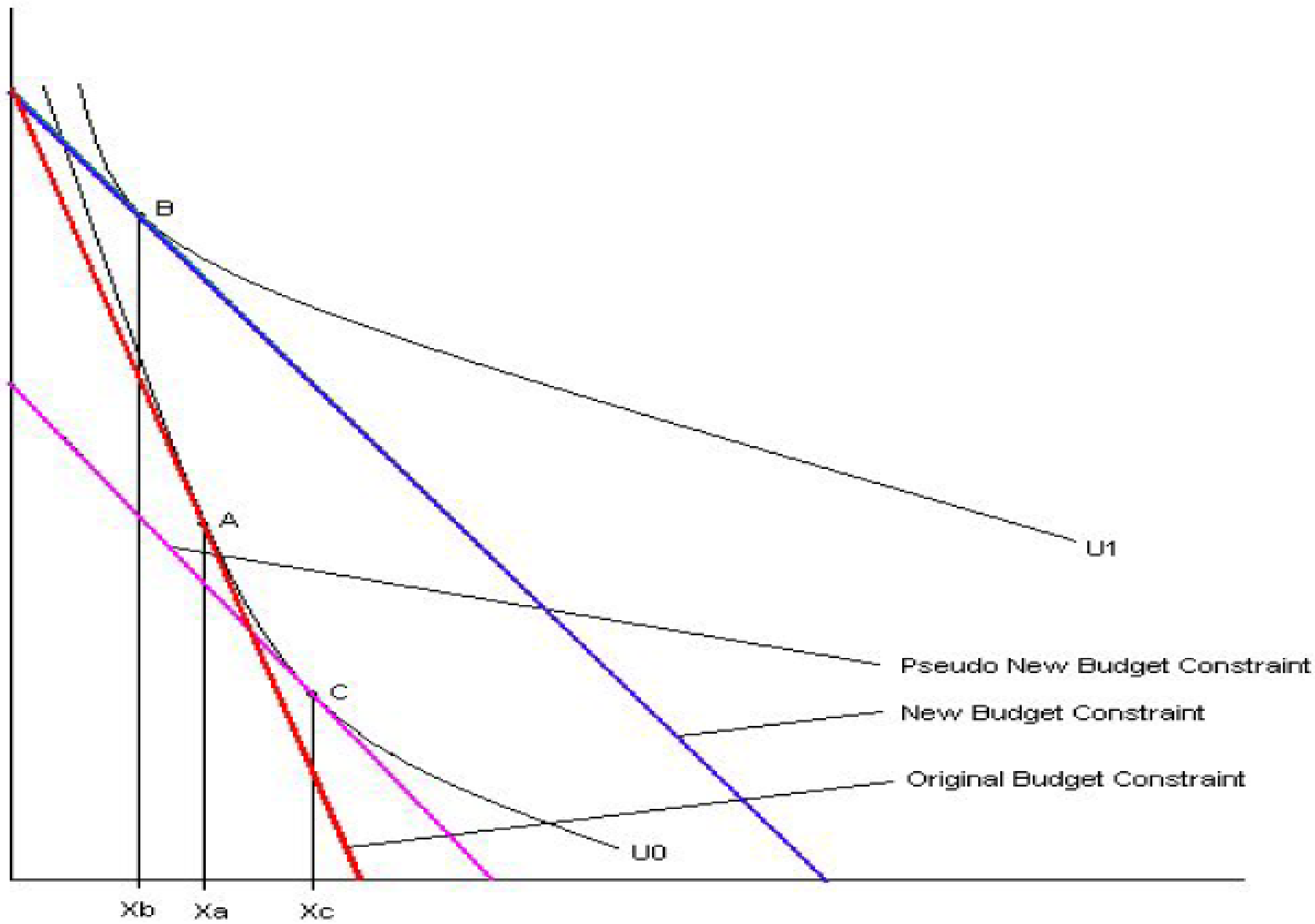
If in addition the utility function is strictly quasi-concave

3. $h_i\,(p,v(p,w)) = x_i(p,w)$
4. $x_i(p,e(p,u)) = h_i(p,u)$

INCOME AND SUBSTITUTION EFFECTS



INCOME AND SUBSTITUTION EFFECTS



THEOREM 1.9 The Slutsky Equation

Let $x(p,w)$ be the consumer's Marshallian demand system. Let u^* be the level of utility the consumer achieves at prices p and income w. Then

$$\underbrace{\frac{\partial x_i(p,w)}{\partial p_j}}_{TE} = \underbrace{\frac{\partial h_i(p,u^*)}{\partial p_j}}_{SE} \underbrace{- x_j(p,w)\frac{\partial x_i(p,w)}{\partial w}}_{IE} \qquad i, j = 1, 2, \ldots n$$

THEOREM 1.10 Negative Own-Substitution Terms

Let $h(p,u)$ *be the Hicksian demand for good i. Then*

$$\frac{\partial h_i(p,u)}{\partial p_i} \leq 0 \qquad i = 1, 2, \ldots n$$

THEOREM 1.11 Properties of the Hicksian demand price derivatives

Let $x^h(p,u)$ *be the consumer's system of Hicksian demands and suppose that the expenditure function* $e(.)$ *is twice continuously differentiable.*

Denote

$$D_p(h(p,u)) = \begin{bmatrix} \dfrac{\partial h_1(p,u)}{\partial p_1} & \dfrac{\partial h_1(p,u)}{\partial p_2} & \cdots & \dfrac{\partial h_1(p,u)}{\partial p_n} \\ \dfrac{\partial h_2(p,u)}{\partial p_1} & \dfrac{\partial h_2(p,u)}{\partial p_2} & \cdots & \dfrac{\partial h_2(p,u)}{\partial p_n} \\ \cdots & & & \\ \dfrac{\partial h_n(p,u)}{\partial p_1} & \dfrac{\partial h_n(p,u)}{\partial p_2} & \cdots & \dfrac{\partial h_n(p,u)}{\partial p_n} \end{bmatrix}$$

Then

1.

$$D_p(h(p,u)) = D^2{}_p(e(p,u)) = \begin{bmatrix} \frac{\partial^2 e(p,u)}{\partial {p_1}^2} & \frac{\partial^2 e(p,u)}{\partial p_2 \partial p_1} & \cdots & \frac{\partial^2 e(p,u)}{\partial p_n \partial p_1} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 e(p,u)}{\partial p_1 \partial p_n} & \frac{\partial^2 e(p,u)}{\partial p_2 \partial p_n} & \cdots & \frac{\partial^2 e(p,u)}{\partial {p_n}^2} \end{bmatrix}$$

2. $D_p(h(p,u))$ *is symmetric* $\Leftrightarrow$ $\frac{\partial h_i(p,u)}{\partial p_j} = \frac{\partial h_j(p,u)}{\partial p_i}$

3. $D_p(h(p,u))$ *is negative semi-definite*

THEOREM 1.12 Symmetric and Negative Semi-definite Slutsky Matrix

Let $x(p,w)$ *be the consumer's Marshallian demand system. Define the ij th Slutsky term as*

$$\frac{\partial x_i(p,w)}{\partial p_j}+\frac{\partial x_i(p,w)}{\partial w}x_j(p,w)$$

and form the entire $n\times n$ ***Slutsky matrix*** *as follows*

$$S(p,w)=\begin{bmatrix} \dfrac{\partial x_1(p,w)}{\partial p_1}+x_1(p,w)\dfrac{\partial x_1(p,w)}{\partial w} & \cdots & \dfrac{\partial x_1(p,w)}{\partial p_n}+x_n(p,w)\dfrac{\partial x_1(p,w)}{\partial w} \\ & \cdots & \\ \cdots & & \\ \dfrac{\partial x_n(p,w)}{\partial p_1}+x_1(p,w)\dfrac{\partial x_n(p,w)}{\partial w} & \cdots & \dfrac{\partial x_n(p,w)}{\partial p_n}+x_n(p,w)\dfrac{\partial x_n(p,w)}{\partial w} \end{bmatrix}$$

Then $S(p,w)$ *is symmetric and negative semi-definite.*